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A FINITE INTEGRAL INVOLVING I-FUNCTION OF TWO VARIABLES

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ABSTRACT

The aim of this research paper is to evaluate a finite integral involving I-function of two variables.

1. INTRODUCTION:

The I–function of two variables introduced by Sharma & Mishra [2], will be defined and represented as follows:

$$I\begin{bmatrix}x\\y\end{bmatrix} = I^{0,n:m_1,n_1:m_2,n_2}_{p_i,q_i:r:p_{i'},q_{i'}:r':p_{i''},q_{i''}:r''}\left[\begin{matrix}x\\y\end{matrix}\middle| \begin{matrix}[(a_j:\alpha_j,A_j)_{1,n}],[(a_{ji}:\alpha_{ji},A_{ji})_{n+1,p_i}]\\ [(b_{ji}:\beta_{ji},B_{ji})_{1,q_i}]\end{matrix} \begin{matrix}:[(c_j;\gamma_j)_{1,n_1}],[(c_{ji'};\gamma_{ji'})_{n_1+1,p_{i'}}];[(e_j;E_j)_{1,n_2}],[(e_{ji''};E_{ji''})_{n_2+1,p_{i''}}]\\ :[(d_j;\delta_j)_{1,m_1}],[(d_{ji'};\delta\gamma_{ji'})_{m_1+1,q_{i'}}];[(f_j;F_j)_{1,m_2}],[(f_{ji''};F_{ji''})_{m_2+1,q_{i''}}]\end{matrix}\right]$$

$$= \frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\phi_1(\xi,\eta)\theta_2(\xi)\theta_3(\eta)\,x^{\xi}y^{\eta}\,d\xi d\eta, \qquad (1)$$

where

$$\phi_1(\xi,\eta) = \frac{\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j\xi+A_j\eta)}{\sum_{i=1}^{r}[\prod_{j=n+1}^{p_i}\Gamma(a_{ji}-\alpha_{ji}\xi-A_{ji}\eta)\prod_{j=1}^{q_i}\Gamma(1-b_{ji}+\beta_{ji}\xi+B_{ji}\eta)},$$

$$\theta_2(\xi) = \frac{\prod_{j=1}^{m_1}\Gamma(d_j-\delta_j\xi)\prod_{j=1}^{n_1}\Gamma(1-c_j+\gamma_j\xi)}{\sum_{i'=1}^{r'}[\prod_{j=m_1+1}^{q_{i'}}\Gamma(1-d_{ji'}+\delta_{ji'}\xi)\prod_{j=n_1+1}^{p_{i'}}\Gamma(c_{ji'}-\gamma_{ji'}\xi)},$$

$$\theta_3(\eta) = \frac{\prod_{j=1}^{m_2}\Gamma(f_j-F_j\eta)\prod_{j=1}^{n_2}\Gamma(1-e_j+E_j\eta)}{\sum_{i''=1}^{r''}[\prod_{j=m_2+1}^{q_{i''}}\Gamma(1-f_{ji''}+F_{ji''}\eta)\prod_{j=n_2+1}^{p_{i''}}\Gamma(e_{ji''}-E_{ji''}\eta)},$$

x and y are not equal to zero, and an empty product is interpreted as unity p_i, $p_{i'}$, $p_{i''}$, q_i, $q_{i'}$, $q_{i''}$, n, n_1, n_2, n_j and m_k are non negative integers such that $p_i \geq n \geq 0$, $p_{i'} \geq n_1 \geq 0$, $p_{i''} \geq n_2 \geq 0$, $q_i > 0$, $q_{i'} \geq 0$, $q_{i''} \geq 0$, (i = 1, …, r; i′ = 1, …, r′; i′′ = 1, …, r′′; k = 1, 2) also all the A's, α's, B's, β's, γ's, δ's, E's and F's are assumed to be positive quantities for standardization purpose; the definition of I-function

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of two variables given above will however, have a meaning even if some of these quantities are zero. The contour L_1 is in the ξ–plane and runs from $-\omega\infty$ to $+\omega\infty$, with loops, if necessary, to ensure that the poles of $\Gamma(d_j-\delta_j\xi)$ (j = 1, ..., m_1) lie to the right, and the poles of $\Gamma(1-c_j+\gamma_j\xi)$ (j = 1, ..., n_1), $\Gamma(1-a_j+\alpha_j\xi+A_j\eta)$ (j = 1, ..., n) to the left of the contour.

The contour L_2 is in the η–plane and runs from $-\omega\infty$ to $+\omega\infty$, with loops, if necessary, to ensure that the poles of $\Gamma(f_j-F_j\eta)$ (j=1,....., n_2) lie to the right, and the poles of $\Gamma(1-e_j+E_j\eta)$ (j = 1, ..., m_2), $\Gamma(1-a_j+\alpha_j\xi+A_j\eta)$ (j = 1, ..., n) to the left of the contour. Also

$$R' = \sum_{j=1}^{p_i}\alpha_{ji} + \sum_{j=1}^{p_i'}\gamma_{ji'} - \sum_{j=1}^{q_i}\beta_{ji} - \sum_{j=1}^{q_i'}\delta_{ji'} < 0,$$

$$S' = \sum_{j=1}^{p_i}A_{ji} + \sum_{j=1}^{p_i''}E_{ji''} - \sum_{j=1}^{q_i}B_{ji} - \sum_{j=1}^{q_i''}F\delta_{ji'} < 0,$$

$$U = \sum_{j=n+1}^{p_i}\alpha_{ji} - \sum_{j=1}^{q_i}\beta_{ji} + \sum_{j=1}^{m_1}\delta_j - \sum_{j=m_1+1}^{q_i'}\delta_{ji'} + \sum_{j=1}^{n_1}\gamma_j - \sum_{j=n_1+1}^{p_i'}\gamma_{ji'} > 0, \tag{2}$$

$$V = -\sum_{j=n+1}^{p_i}A_{ji} - \sum_{j=1}^{q_i}B_{ji} - \sum_{j=1}^{m_2}F_j - \sum_{j=m_{21}+1}^{q_i''}F_{ji''} + \sum_{j=1}^{n_2}E_j - \sum_{j=n_2+1}^{p_i''}E_{ji''} > 0, \tag{3}$$

and $|\arg x| < \frac{1}{2}U\pi$, $|\arg y| < \frac{1}{2}V\pi$.

In our investigation we shall need the following result:

From Bhonsle [1, p. 91 Eq.(3.1)], we have

$$\int_0^1 x^{\sigma-\lambda-\delta}(1-x^2)^{-\frac{1}{2}\mu}P_\nu^\mu(x)J_\delta(xt)J_\lambda(xt)dx$$
$$= \frac{2^{\mu-\lambda-\delta-1}\Gamma\left(\frac{1}{2}+\frac{1}{2}\sigma\right)\Gamma(1+\frac{1}{2}\sigma)t^{\lambda+\delta}}{\Gamma(\delta+1)\Gamma(\lambda+1)\Gamma\left(1+\frac{1}{2}\sigma-\frac{1}{2}\nu-\frac{1}{2}\mu\right)\Gamma(\frac{3}{2}+\frac{1}{2}\sigma+\frac{1}{2}\nu-\frac{1}{2}\mu)}$$
$$\times {}_4F_5\left[{\frac{1}{2}+\frac{1}{2}\sigma, 1+\frac{1}{2}\sigma, \frac{1}{2}(\lambda+\delta+1), \frac{1}{2}(\lambda+\delta+2); \atop 1+\frac{1}{2}\sigma-\frac{1}{2}\nu-\frac{1}{2}\mu, \frac{3}{2}+\frac{1}{2}\sigma+\frac{1}{2}\nu-\frac{1}{2}\mu, \delta+1, \lambda+1, \lambda+\delta+1;} -t^2\right], \tag{4}$$

provided that Re $(\mu) < 1$, Re$(\sigma) > -1$.

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2. FINITE INTEGRAL:

In this section, we shall establish following integral:

$$\int_0^1 x^{\sigma-\lambda-\delta}(1-x^2)^{-\frac{1}{2}\mu} P_\nu^\mu(x) J_\delta(xt) J_\lambda(xt)\; I^{0,\ n\ :m_1,n_1\ :m_2,n_2}_{p_i,q_i:r\ :p_i',q_i':r'\ :p_i'',q_i'':r''}\begin{bmatrix} z_1 x^\rho \\ z_2 \end{bmatrix} dx$$

$$= \frac{2^{\mu-\lambda-\delta-1} t^{\lambda+\delta}\sqrt{\pi}}{\Gamma(\delta+1)\Gamma(\lambda+1)} . \sum_{r=0}^{\infty} \frac{2^{-2r}\left(\frac{\lambda+\delta+1}{2}\right)_r \left(\frac{\lambda+\delta+2}{2}\right)_r (-t^2)^r}{(\lambda+1)_r(\delta+1)_r(\lambda+\delta+1)_r\, r!}\; I^{0,\ n\ :m_1,n_1+1\ :m_2,n_2}_{p_i,q_i:r\ :p_i'+1,q_i'+2:r'\ :p_i'',q_i'':r''}\left[\begin{matrix} z_1 2^{-\rho} \\ z_2 \end{matrix}\,\middle|\, \begin{matrix} \ldots\ldots,\ldots:(-\sigma-2r,\rho),\ldots,\ldots:\ldots,\ldots \\ \ldots\ldots,\ldots:\ldots,\ldots,\left(-\frac{\sigma}{2}+\frac{\nu}{2}+\frac{\mu}{2}-r,\frac{\rho}{2}\right),\left(-\frac{1}{2}-\frac{\sigma}{2}-\frac{\nu}{2}+\frac{\mu}{2}-r,\frac{\rho}{2}\right):\ldots,\ldots \end{matrix}\right], \qquad (5)$$

provided that $\mathrm{Re}(\mu) < 1$, $\mathrm{Re}(\sigma) > -1$, $\mathrm{Re}(\sigma+1) + \rho \min_{1\leq j\leq m} \mathrm{Re}\left(-\frac{b_j}{\beta_j}\right) > 0$, $|\arg z_1| < \frac{1}{2} U\pi$, $|\arg z_2| < \frac{1}{2} V\pi$, where U and V is given in (2) and (3) respectively.

Proof:

To establish (5), replace the I-function by its equivalent counter integral as given in (1), we get

$$\int_0^1 x^{\sigma-\lambda-\delta}(1-x^2)^{-\frac{1}{2}\mu} P_\nu^\mu(x) J_\delta(xt) J_\lambda(xt)$$

$$. \left[\frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2} \phi_1(\xi,\eta)\,\theta_2(\xi)\theta_3(\eta) z_1^{\xi} z_2^{\eta}\, d\xi d\eta\right] dx$$

Change the order of integration, which is valid under the given condition, we arrive at

$$\frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2} \phi_1(\xi,\eta)\,\theta_2(\xi)\theta_3(\eta) z_1^{\xi} z_2^{\eta}$$

$$.\left[\int_0^1 x^{\sigma+\rho\xi-\lambda-\delta}(1-x^2)^{-\frac{1}{2}\mu} P_\nu^\mu(x) J_\delta(xt) J_\lambda(xt)\, dx\right] d\xi d\eta.$$

Now evaluate the inner integral with the help of (4), we get

$$\frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2} \phi_1(\xi,\eta)\,\theta_2(\xi)\theta_3(\eta) z_1^{\xi} z_2^{\eta}$$

$$\times \frac{2^{\mu-\lambda-\delta-1}\Gamma\left[\frac{1}{2}+\frac{1}{2}(\sigma+\rho\xi)\right]\Gamma[1+\frac{1}{2}(\sigma+\rho\xi)]t^{\lambda+\delta}}{\Gamma(\delta+1)\Gamma(\lambda+1)\Gamma\left[1+\frac{1}{2}(\sigma+\rho\xi)-\frac{1}{2}\nu-\frac{1}{2}\mu\right]\Gamma[\frac{3}{2}+\frac{1}{2}(\sigma+\rho\xi)+\frac{1}{2}\nu-\frac{1}{2}\mu]}$$

$$\times {}_4F_5\left[{\frac{1}{2}+\frac{1}{2}(\sigma+\rho\xi),1+\frac{1}{2}(\sigma+\rho\xi),\frac{1}{2}(\lambda+\delta+1),\frac{1}{2}(\lambda+\delta+2); \atop 1+\frac{1}{2}(\sigma+\rho\xi)-\frac{1}{2}\nu-\frac{1}{2}\mu,\frac{3}{2}+\frac{1}{2}(\sigma+\rho\xi)+\frac{1}{2}\nu-\frac{1}{2}\mu,\delta+1,\lambda+1,\lambda+\delta+1;} - t^2\right]d\xi d\eta$$

Now representing the function ${}_4F_5$ involved in the above expression in a series form and using the duplication formula for the Gama function, we easily get after a little simplification:

$$\frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\phi_1(\xi,\eta)\,\theta_2(\xi)\theta_3(\eta)z_1{}^{\xi}z_2{}^{\eta}$$
$$\times\left[\frac{2^{\mu-\lambda-\delta-\sigma-1}t^{\lambda+\delta}\sqrt{\pi}}{\Gamma(\delta+1)\Gamma(\lambda+1)}.\sum_{r=0}^{\infty}\frac{2^{-2r}\left(\frac{\lambda+\delta+1}{2}\right)_r\left(\frac{\lambda+\delta+2}{2}\right)_r}{(\lambda+1)_r(\delta+1)_r(\lambda+\delta+1)_r}\right.$$
$$\left.\cdot\frac{(-t^2)^r}{\Gamma\left(1+\frac{1}{2}\sigma-\frac{1}{2}\nu-\frac{1}{2}\mu+r+\frac{1}{2}\rho\xi\right)\Gamma\left(\frac{3}{2}+\frac{1}{2}\sigma+\frac{1}{2}\nu-\frac{1}{2}\mu+r+\frac{1}{2}\rho\xi\right)r!}\right]d\xi d\eta$$

Changing the order of integration and summation in above expression and interpreting the result thus obtained with the help of (1), we get the required result (5).

3. PARTICULAR CASES:

I. On specializing the parameters in main integral, we get following integral in terms of I-function of one variable:

$$\int_0^1 x^{\sigma-\lambda-\delta}\,(1-x^2)^{-\frac{1}{2}\mu}P_\nu^\mu(x)J_\delta(xt)J_\lambda(xt)I_{p_i,q_i:r}^{m,n}[zx^\rho]\,dx$$

$$=\frac{2^{\mu-\lambda-\delta-1}t^{\lambda+\delta}\sqrt{\pi}}{\Gamma(\delta+1)\Gamma(\lambda+1)}.\sum_{r=0}^{\infty}\frac{2^{-2r}\left(\frac{\lambda+\delta+1}{2}\right)_r\left(\frac{\lambda+\delta+2}{2}\right)_r(-t^2)^r}{(\lambda+1)_r(\delta+1)_r(\lambda+\delta+1)_r r!}$$

$$.\,I_{p_i+1,q_i+2:r}^{m,n+1}\left[z2^{-\rho}\left|{(-\sigma-2r,\rho),\ldots\ldots,\ldots\ldots \atop \ldots\ldots,\ldots\ldots,\left(-\frac{\sigma}{2}+\frac{\nu}{2}+\frac{\mu}{2}-r,\frac{\rho}{2}\right),\left(-\frac{1}{2}-\frac{\sigma}{2}-\frac{\nu}{2}+\frac{\mu}{2}-r,\frac{\rho}{2}\right)}\right.\right], \qquad (6)$$

provided that $Re(\mu) < 1$, $Re(\sigma) > -1$, $Re(\sigma+1) + \rho\min_{1\leq j\leq m} Re(-\frac{b_j}{\beta_j}) > 0$, $|\arg z| < \frac{1}{2}\pi B$, where B is given given by $B = \sum_{j=1}^{n}\alpha_j - \sum_{j=n+1}^{p_i}\alpha_{ji} + \sum_{j=1}^{m}\beta_j - \sum_{j=m+1}^{q_i}\beta_{ji} > 0$.

II. On choosing r = 1 in the integral (6), we get following integral in terms of H-function of one variable:

$$\int_0^1 x^{\sigma-\lambda-\delta}\,(1-x^2)^{-\frac{1}{2}\mu} P_\nu^\mu(x) J_\delta(xt) J_\lambda(xt) H_{p,q}^{m,n}[zx^\rho]\,dx$$

$$= \frac{2^{\mu-\lambda-\delta-1} t^{\lambda+\delta}\sqrt{\pi}}{\Gamma(\delta+1)\Gamma(\lambda+1)} . \sum_{r=0}^{\infty} \frac{2^{-2r}\left(\frac{\lambda+\delta+1}{2}\right)_r \left(\frac{\lambda+\delta+2}{2}\right)_r (-t^2)^r}{(\lambda+1)_r(\delta+1)_r(\lambda+\delta+1)_r r!}$$

$$. H_{p+1,q+2}^{m,n+1}\left[z2^{-\rho}\left|{}^{(-\sigma-2r,\rho),(a_j,\alpha_j)_{1,p}}_{(b_j,\beta_j)_{1,q},\left(-\frac{\sigma}{2}+\frac{\nu}{2}+\frac{\mu}{2}-r,\frac{\rho}{2}\right),\left(-\frac{1}{2}-\frac{\sigma}{2}-\frac{\nu}{2}+\frac{\mu}{2}-r,\frac{\rho}{2}\right)}\right.\right], \qquad (7)$$

provided that $Re(\mu) < 1$, $Re(\sigma) > -1$, $Re(\sigma+1) + \rho \min_{1\le j\le m} Re(-\frac{b_j}{\beta_j}) > 0$, $|\arg z| <$ ½πA, where A is given by $A = \sum_{j=1}^{n}\alpha_j - \sum_{j=n+1}^{p}\alpha_j + \sum_{j=1}^{m}\beta_j - \sum_{j=m+1}^{q}\beta_j > 0$.

REFERENCES

1. Bhonsle, B. R.: Some on finite integral involving the product of Whittaker functions and generalized hypergeometric functions, Bull. Calcutta Math. Soc. 49 (1957), 89-93.
2. Sharma C. K. and Mishra, P. L.: On the I-function of two variables and its certain properties, ACI, 17 (1991), 1-4.